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Consistency & Stability Issues in the Numerical Integration of the First & Second Order Initial Value Quandary

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Abstract

In this note we consider some basic, yet unusual, issues pertaining to the accuracy and stability of numerical integration methods and procedures to follow the solution of first order and second order initial value problems (IVP) and quandaries. Included are remarks on multiple solutions, multi-step methods and procedures, effect of initial value perturbations, as well as slowing and advancing the computed motion in second order problems and quandaries.

Keywords: *Consistency, Initial Value Problems, Numerical Integration, Sensitivity to Initial Conditions, Slowing and Advancing the Computed Motion.*

1. Introduction

Numerically following the solution of an evolution equation in time is one of the central tasks of numerical analysis and has received over the year's vast and repeated attention, see [1] [2]. In this note we consider some essential and interesting, but possibly not widely known, aspects of the numerical solution of initial value problems and quandaries. The rationale behind the procedures advocated here for the accurate and stable solution of the initial value problem and quandary, is often to greatly facilitate their introduction in the classroom. For the first order problem and quandary, we consider the usefulness of implicit methods and procedures, see [3], for capturing multiple solutions emanating from a point of bifurcations, and also simple procedures for determining the accuracy and stability of multistep methods and procedures. For the second order problem and quandary, see [4], we consider the means of slowing and advancing the computed speed of the numerical solution of the equation of motion.

2. The Generalized Mean Value Theorem—Taylor's Theorem

The decisive significance of Taylor's theorem (as can be looked up in any elementary calculus textbook) to applied mathematics in general, and to numerical analysis in particular, is that it ascertains that every differential function looks locally like a polynomial. Polynomials having the advantage of being easily computed, differentiated and integrated, relieving us by their use of the burden of possibly high complications in the symbolic manipulation of functions, often only implicitly given as the solution of an initial value problem (IVP) or a boundary value problem (BVP). We look at this theorem here from an unusual angle.

The mean value theorem (MVT) states that if function f(x), f (0) =0,$f'(0) \neq 0$, is continuous in the closed interval [0, x] and differentiable on the open interval (0, x), then point ξ exists, strictly inside the interval, $0 < \xi < x$, which the slope of the chord equals to the slope of the tangent line to f(x) at point ξ, or

$$\frac{f(0)-f(0)}{x} = f'(\xi), \text{ or } f(0) = xf'(\xi), 0 < \xi < x \quad (1)$$

Implying that f(x) is nearly linear near x = 0. The MVT theorem is a direct result, of the geometrically intuitively plausible, Rolle’s Theorem. The generalized mean value theorem (GMVT) is a result of the application of the MVT (or Rolle’s Theorem) to the higher order derivative functions of f(x). Here it is in its most concise form: Let function f(x) be continuous at x = 0, and such that

$$f(0) = 0, f'(0) = 0, f''(0) = 0, \ldots, f^{(n-1)}(0) = 0, f^{(n)}(0) \neq 0 \quad (2)$$

Then, function f(x) may be expressed in the form

$$f(x) = \frac{1}{n!}x^n f^{(n)}(\xi), 0 < \xi < x \quad (3)$$

Implying that if $f^{(n)}(x)$ is bounded near x = 0, then f(x), like x^n, is small if $|x| \ll 1$.

2.1 Where is ξ

We consider first the simplest case of n = 1, for which Equation (3) is

$$f(x) = xf'(\xi), f(0) = 0, f'(0) \neq 0, 0 < \xi < x \quad (4)$$

$$f(x) = Ax + Bx^2 + Cx^3, f'(x) = A + 2Bx + 3Cx^2 \quad (5)$$

Such that $A = f'(0), B = f''(0)/2!, C = f'''(0)/3!$. We further assume that, approximately

$$\xi = kx + mx^2 \quad (6)$$

And have from this that

$$f(x) - xf'(\xi) = B(1-2k)x^2 + (G - 3Gk^2 - 2Bm)x^3 + O(x^4) \quad (7)$$

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Annulling the first two terms of the above equation results in

$$k = \frac{1}{2}, m = \frac{1}{8} * \frac{G}{B}, \text{or } m = \frac{1}{24} * \frac{f'''(0)}{f''(0)} \quad (8)$$

Or generally, for any n in Equation (2)

$$\xi = \frac{1}{n+1} x, \text{ if } |x| \ll 1 \quad (9)$$

3. Polynomial Approximations

Some examples will convince us of the decisive usefulness of the GMVT theorem to numerical analysis. As a first example, we use the theorem to get a good polynomial approximation to e^x near x = 0. We start by writing

$$r(x) = e^x - (a + bx) \quad (10)$$

And propose to fix free parameters a and b such that $r(0) = 0$, $r'(0) = 0$, namely, such that

$$r(x) = e^x - (1 + x), r''(x) = e^x, r''(0) = 1 \neq 0 \quad (11)$$

Now, by fundamental Equation (3), we may write r(x) of $r(0) = 0$, $r'(0) = 0$ as

$$r(x) = \frac{1}{2} x^2 r''(\xi) \text{ or } e^x = 1 + x + \frac{1}{2} x^2 e^{\xi}, () < \xi < x \quad (12)$$

Providing us with a good linear polynomial approximation to e^x in the vicinity of $x = 0$.

Moreover, since e^x is an increasing function, we readily obtain from Equation (12) the strict inequalities

$$1 + x + \frac{1}{2}x^2 < e^x < 1 + x + \frac{1}{2}x^2 e^x \quad (13)$$

$$1 + x + \frac{1}{2}x^2 < e^x < \frac{1+x}{1-\frac{1}{2}x^2} \quad (14)$$

3.1 Improving the Polynomial Approximation with ξ

The reason the lower bound on e^x in the above inequality (14) is better then the upper bound is due to the fact that as the order of the approximation increases, ξ moves rather ever closer to the osculation point x = 0.Here, since r(0) =0, $r'(0) = 0$, $r''(0) \neq 0$, then, according to Equation (9), $\xi = \frac{x}{3}$, nearly, if $|x| \ll 1$.

Replacing e^{ξ} in Equation (12) by $1+\xi$ with $\xi=\frac{x}{3}$ we obtain, forthwith, the better approximation

$$e^{x}=1+x+\frac{1}{2}x^{2}(1+\xi)=1+x+\frac{1}{2}x^{2}\left(1+\frac{1}{3}x\right)=1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3} \tag{15}$$

Otherwise we may start from

$$e^{x}=1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}e^{\xi},\ 0<\xi<x \tag{16}$$

Take $\xi=kx+mx^{2}$, expand further

$$e^{\xi}=1+kx+\left(\frac{1}{2}k^{2}+m\right)x^{2}+\left(\frac{1}{6}k^{3}+km\right)x^{3}+\left(\frac{1}{24}k^{4}+\frac{1}{2}k^{2}m+\frac{1}{2}m^{2}\right)x^{4}+\cdots \tag{17}$$

$$e^{\xi}=1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{k}{6}x^{4}+\left(\frac{1}{12}k^{2}+\frac{m}{6}\right)x^{5}+\left(\frac{1}{36}k^{3}+\frac{km}{6}\right)x^{6}+\cdots \tag{18}$$

To have in the above equation as many correct terms as possible we set

$$\frac{k}{6}=\frac{1}{4!}\quad\text{and}\quad\left(\frac{1}{12}k^{2}+\frac{m}{6}\right)=\frac{1}{5!} \tag{19}$$

Resulting in

$$\xi=\frac{1}{4}x+\frac{3}{160}x^{2}\quad\text{if } x\ll 1 \tag{20}$$

3.2 Trigonometric Function Approximations

Taylor's theorem is not restricted to polynomials, but may be advantageously used to directly construct other approximating functions. For instance, we may start with

$$r(x)=\sqrt{1+x}-(a\mathrm{Cos}(x)+b\mathrm{Sin}(x)) \tag{21}$$

And use free numbers a and b to enforce $r(0)=0$ and $r'(0)=0$, to have

$$r(x)=\sqrt{1+x}-\left(\mathrm{Cos}(x)+\frac{1}{2}\mathrm{Sin}(x)\right)$$

$$r''(x)=-\frac{1}{4}(1+x)^{-\frac{3}{2}}+\mathrm{Cos}(x)+\frac{1}{2}\mathrm{Sin}(x),\ r''(0)=\frac{3}{4}\neq 0 \tag{22}$$

Consequently, the Taylor, or the GMVT, form of r(x) is

$$r(x) = \frac{1}{2}x^2 r''(\xi) = \frac{1}{2}x^2\left(-\frac{1}{4}(1+\xi)^{-\frac{3}{2}} + \mathrm{Cos}(\xi) + \frac{1}{2}\mathrm{Sin}(\xi)\right), 0 < \xi < x \quad (23)$$

or, asymptotically, as $x \to 0, \xi \to 0$.

$$\sqrt{1+x} = \mathrm{Cos}(x) + \frac{1}{2}\mathrm{Sin}(x) + \frac{3}{8}x^2, \text{ if } |x| \ll 1 \quad (24)$$

3.3 Rational Function Approximations

Rational approximations are also desirable, and efficient. Here we start with, say

$$r(x) = e^x - \frac{a+bx}{1+cx} \quad (25)$$

Of the three free parameters a, b, c. imposing on r(x) the conditions

$$r(0) = 0, r'(0) = 0, r''(0) = 0 \quad (26)$$

$$e^x = \frac{2+x}{2-x} - \frac{1}{12}x^3, \text{ If } x \ll 1 \quad (27)$$

If the point of osculation is not $x = 0$, but $x = a$, then x in the theorem is shifted to $x - a$.

4. First Order Linear Homogeneous Recursions

In the numerical integration of IVPs we are constantly confronted by the need to solve linear homogeneous recursions. The first order, homogeneous, recursion

$$y_{n+1} + by_n = 0, n = 0, 1, 2, \ldots \quad (28)$$

Where b is a constants independent of n, is brought, without much ado, to the explicit, closed-form, representation

$$y_n = (-b)^n y_n \quad (29)$$

by merely repeating the recursion. We note that if $|b| > 1$, then y_n keeps growing with n, while if $|b| < 1$, then $y_n \to 0$, as $n \to \alpha$.

4.1 Second Order Linear Homogeneous Recursions

Next we consider the three-term homogeneous recursion. Let the sequence $y_0, y_1, y_2, \ldots, y_{n-1}, y_n, y_{n+1}$ be generated by the homogeneous recursion

$$y_{n+2} + by_{n+1} + cy_n = 0 \tag{30}$$

with coefficients b and c assumed independent of n. Recursion (30) is satisfied by $y_n = z^n$ provided z is a root the characteristic equation

$$z^2 + bz + c = 0 \tag{31}$$

In case the two roots z_1, z_2 of Equation (31) are distinct, $z_1 \neq z_2$, then by the linearity of the recursion we have the general solution of this recursion in the form

$$y_n = c_1 z_1^n + c_2 z_2^n \tag{32}$$

with c_1 and c_2 determined by the initial conditions y_0 and y_1. In case the roots of Equation (31) are equal, $z_1 = z_2 = z = -\frac{b}{2}$, $b^2 - 4c = 0$, then we verify that $y_n = c_1 z^n + c_2 n z^n$, with

$$c_1 = y_0 \text{ , } c_2 = \frac{y_1}{1+y_0} \tag{33}$$

In case the roots of Equation (31) are complex conjugates, $z_1 = \alpha + i\beta, z_2 = \alpha - i\beta$, $z_1 z_2 = |z|^2 = c = \alpha^2 + \beta^2$, then we may put z in the form

$$z = |z|e^{\pm i\theta}, z = |z|(\text{Cos}(\theta) \pm i\,\text{Sin}(\theta)), z^n = |z|^n\ (\text{Cos}(n\theta) \pm i\,\text{Sin}(n\theta)) \tag{34}$$

Where $\text{Cos}(\theta) = \alpha/|z|, \text{Sin}(\theta) = \beta/|z|$. Now, $y_n = c_1 z_1^n + c_2 z_2^n$ becomes

$$y_n = |z|^n\big((c_1 + c_2)\text{Cos}(n\theta) + i\ (c_1 - c_2)\text{Sin}(n\theta)\big) \tag{35}$$

With c_1, c_2 fixed by the initial conditions y_0, y_1. For example, the recursion

$$y_2 - 3y_1 + 2y_0 = 0,\ y_0 = 1\ , y_1 = 1{+}\epsilon \tag{36}$$

Results
$$y_n = 1{+}\epsilon\ (2^n - 1) \tag{37}$$

5. The Advantage of an Implicit Method at a Branching Point

Implicit methods for the numerical integration of the first-order IVP hold some stability advantages, but they may require the solution of a nonlinear equation for the next predicted value. At a point of bifurcation they hold the extra advantage of capturing multiple solutions, otherwise missed by an explicit method. Here is an example. The initial value problem

$$y' = -\sqrt{1-y^2}\,, y(0) = 1 \quad , y'(0) = 0 \tag{38}$$

Is solved by both

$$y(t) = 1 \quad \text{and} \quad y(t) = \text{Cos}\,(t) \tag{39}$$

Using the Euler explicit method

$$y_1 = y_0 + \tau y_0' \tag{40}$$

Where y_1 is an approximation to y (τ), we have

$$y_1 = y_0 \text{ and } y_n = 1 \tag{41}$$

Which is only the first solution y (t) = 1 of IVP (38). Using the implicit method

$$y_1 = y_0 + \frac{1}{2}\tau(y_0' + y_1') \tag{42}$$

$$y_1 = 1 - \frac{1}{2}\tau\sqrt{1-y_1^2} \tag{43}$$

We obtain

then the quadratic equation

$$\left(1 + \frac{1}{4}\tau^2\right)y_1^2 - 2y_1 + \left(1 - \frac{1}{4}\tau^2\right) = 0 \tag{44}$$

For y_1, solved by

$$\text{A first } \; y_1 = 1, \text{ and a second } y_1 = \frac{4-\tau^2}{4+\tau^2} = 1 - \frac{1}{2}\tau^2 + \frac{1}{8}\tau^4 + O(\tau^6) \tag{45}$$

As compared with

$$y(\tau) = \text{Cos}(\tau) = 1 - \frac{1}{2}\tau^2 + \frac{1}{24}\tau^4 + O(\tau^6) \tag{46}$$

6. Sensitivity to Initial Conditions

The solution of the IVP

$$y' = y - 1\,, y(0) = y_0 \tag{47}$$

$$y(t) = 1 + (y_0 - 1)e^{t} \tag{48}$$

And if $y_0 = 1$, then $y(t) = 1$, but if $y_0 > 1$, then $y(t) \to \alpha$ as $t \to \alpha$.

7. Determination of the Order of Consistency of Multistep Methods and Procedures

To fully, yet concisely, demonstrate the consistency and stability issues in the integration of first order IVP, and their resolution, we shall look in detail at the general two-step method

$$y_2 = \alpha_0 y_0 + \alpha_1 y_1 + \tau(\beta_0 y_0' + \beta_1 y_1') + err\ ,\ \ y - y(t) \tag{49}$$

In which y_2 is the computed approximation to the correct y(2τ), in which err is the error $y(2\tau) - y_2$, and in which α_0, α_1, β_0, β_1 are free parameters to be determined for highest accuracy and method stability.

In accordance with Taylor's theorem we require, for the highest possible order of consistency, that the calculated y_2 is the correct y(2τ), namely err = 0, for

$$y = 1,\ y = t,\ y = t^2 \tag{50}$$

Leading to the system of equations

$$\alpha_0 + \alpha_1 = 1\ , \alpha_1 + \beta_0 + \beta_1 = 2\ ,\ \alpha_1 + 2\beta_1 = 4 \tag{51}$$

$$\alpha_1 = 1 - \alpha_0\ ,\ \beta_0 = \frac{1}{2}(-1 + \alpha_0),\ \beta_1 = \frac{1}{2}(3 + \alpha_0) \tag{52}$$

In which we leave α_0 free for now, to use it next to guarantee the stability of the method. According to Taylor's theorem the worst case error arises from the next order polynomial, or the function with a constant third derivative. Accordingly, we take next

$$y(t) = \frac{1}{6} M_3 t^3\ , y'''(t) = M_3 \tag{53}$$

And have from Equation (49) and Equation (51) that

$$y(2t) - y_2 = \frac{5-a_0}{12} M_3 t^3 \tag{54}$$

Which is the local error per step? After n steps the error rises to the global $O(t^2)$.

8. Stability of the Multistep Method

The following Lemma is greatly useful for ascertaining the stability of an integration method.

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Lemma. Let real roots z(τ) of the characteristic equation for the integration scheme of the first-order IVP be such that z (τ =0) ≤1 . Then, a sufficient condition for the (conditional) stability of the method is that at z (τ =0) =1, $\frac{dz}{d\tau} < 0$, and that at z (τ =0) = -1 , $\frac{dz}{d\tau} > 0$, Proof. It results directly from the continuity and differentiability of z(τ) that if z (τ =0) =1 and $\frac{dz}{d\tau} < 0$ at $\tau = 0$, then $z(\tau) < 1$ for some $\tau > 0$. See also [5]. Specifically, for the model IVP

$$y' = -y, y(0) = y_0 = 1, y(\tau) = y(0)e^{-\tau} \tag{55}$$

The two-step method of Equation (49) becomes

$$2y_2 = (2\alpha_0 + \tau(1-\alpha_0))y_0 + (2 - 2\alpha_0 - \tau(3+\alpha_0))y_1 \tag{56}$$

Of the characteristic equation

$$2z^2 + \left(-2 + 2\alpha_0 + \tau(3+\alpha_0)\right)z + \left(-2\alpha_0 + \tau(-1+\alpha_0)\right) = 0 \tag{57}$$

At τ = 0 the equation reduces to

$$z^2 + (-1+\alpha_0)z - \alpha_0 = 0 \tag{58}$$

Which is of the two roots ,

$$z_1 = 1, z_2 = -\alpha_0, \text{ and } -1 < \alpha_0 \le 1 \tag{59}$$

To verify the stability of the method we seek z ′(τ) at τ = 0. Implicitly differentiating characteristic Equation (57) with respect to τ we have

$$4zz' + (3+\alpha_0)z + \left(-2 + 2\alpha_0 + \tau(3+\alpha_0)\right)z' - 1 + \alpha_0 = 0 \tag{60}$$

At τ = 0 , the above equation reduces to

$$4zz' + (3+\alpha_0)z + (-2+2\alpha_0)z' - 1 + \alpha_0 = 0 \tag{61}$$

And for $z_1 = 1$ we obtain from the above equation that

$$z'(\tau = 0) = -\frac{2(1+\alpha_0)}{2+\alpha_0} < 0 \tag{62}$$

And since for stability z_1 (τ) needs to come down at τ = 0, hence $\alpha_0 > -1$.

9. A Three Step Method

The integration method

$$y_3 = y_2 + \frac{1}{12}\tau(23y_2' - 16y_1' + 5y_0') \tag{63}$$

Is correct for $y = 1$, $y = \tau$, $y = \tau^2$ and $y = \tau^3$. For

$$y(\tau) = \frac{1}{24}M_4\tau^4 \tag{64}$$

We have from Equation (63) that

$$y(3\tau) - y_3 = \frac{3}{8}M_4\tau^3 \tag{65}$$

For $y' = -y$, $y(0) = y_0$ the characteristic equation of the three step method becomes

$$12z^3 + (-12 + 23\tau)z^2 - 16\tau z + 5\tau = 0 \tag{66}$$

And at $\tau = 0$ it reduces to

$$12z^3 - 12z^2 = 0, \text{ of roots } z_1 = 1, z_2 = z_3 = 0 \tag{67}$$

Implicit differentiation of Equation (66) with respect to τ yields

$$36z^2z' + 23z^2 + (-12 + 23\tau)2zz' - 16z - 16\tau z' + 5 = 0 \tag{68}$$

And at $\tau = 0$, $z = 1$, we have that $z' = -1$, so that near $\tau = 0$

$$z = 1 - \tau \tag{69}$$

And the method is stable. We further have from Equation (67) that at $\tau = 6/11$, $z_3 = -1$ and $z_1 = z_2 = 1/2$. At the repeating root $z = 0$, the derivative function z' does not exist. Instead we write τ in terms of z as

$$\tau = \frac{12(1-z)}{23z^2 - 12z + 5}z^2 \tag{70}$$

And if $z = 0$, nearly, then

$$\tau = \frac{12}{5}z^2 \tag{71}$$

$$z = \pm\sqrt{\frac{5}{12}\tau} \tag{72}$$

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10. Advancing and Retarding the Computed Motion

Next, we turn our attention to the second order IVP, see [6] and [7], as typified by the model second order problem

$$y'' + y = 0 \text{ , } y(0) = y_0 \text{ , } y'(0) = y_0' \tag{73}$$

Which we propose to approximate as

$$\frac{y_0 - 2y_1 + y_2}{\tau^2} + (1+\epsilon)y_1 = 0 \text{ , } \epsilon = \alpha\tau + \beta\tau^2 \tag{74}$$

The characteristic equation of this method and procedure is

$$z^2 + \left(-2 + \tau^2(1+\epsilon)\right)z + 1 = 0 \tag{75}$$

For a sufficiently small τ, the roots of the characteristic equation are complex and $|z| = 1$. Hence the closed-form prediction of the computed y at step n

$$y_n = c_1 \text{Cos}(n\theta) + c_2 \text{Sin}(n\theta) \tag{76}$$

Where c_1 and c_2 are determined by the initial conditions. From the characteristic Equation (75) we have that

$$\text{Cos}(\theta) = 1 - \frac{1}{2}\tau^2(1+\epsilon), \text{ Sin}(\theta) = \frac{1}{2}\sqrt{4 - (-2 + \tau^2(1+\epsilon))^2} \tag{77}$$

$$\theta = \tau + \frac{1}{2}\alpha\tau^2 + \frac{1}{24}(1 - 3\alpha^2 12\beta)\tau^3 + O(\tau^5) \text{ , } \epsilon = 2 - \tau^2 - 2\text{Cos}(\tau) \tag{78}$$

To drop the τ^2 term in the above equation we take α = 0, and are left with

$$\frac{\theta}{\tau} = 1 + \frac{1}{24}(1 + 12\beta)\tau^2 + O(\tau^4) \tag{79}$$

Suggesting that θ may be advanced or retarded relative to τ with a proper choice of β. For instance, for β = −1/12 we have that θ = τ, nearly.

11. Period Control

A cycle is completed when $\sin(n\theta) = 0$ or $n\theta = 2\pi$. Then, according to Equation (108) $y_n = 0$ and $x_n = 1$. From $\tau n\theta = 2\pi\tau$ and $T = n\tau$ we obtain the computed period as

$$T = 2\pi\frac{\tau}{\theta} \tag{80}$$

and to retain $T = 2\pi$ we select α_1 in Equation (93) so as to guarantee $\tau = \theta$ or $\text{Sin}(\theta)$. This condition becomes, in view of Equation

$$\text{Sin}(\tau) = \tau\sqrt{\alpha_1 - \frac{1}{4}\alpha_1^2\tau^2} \tag{81}$$

Leading to the quadratic equation

$$\frac{1}{4}\alpha_1^2\tau^2 - \alpha_1 + \frac{\text{Sin}^2(\tau)}{\tau^2} = 0 \tag{82}$$

For α_1, and resulting in

$$\alpha_1 = \frac{2(1-\cos(\tau))}{\tau^2} \tag{83}$$

$$\alpha_1 = 1 - \frac{1}{12\tau^2} + \frac{1}{360\tau^4} \tag{94}$$

if τ is small.

12. Quadratic Prediction

Inclusion of the acceleration in the prediction of x_1 suggests the higher order scheme

$$x_1 = x_0 + \tau x_0' + \frac{1}{2}\tau^2 x_0''$$

$$y_1 = y_0 + \frac{1}{2}\tau(\alpha_0 y_0' + \alpha_1 y_1') \tag{85}$$

That becomes for

$$x' = -y \quad , y' = x \; , \qquad x'' = -x$$

$$x_1 = x_0 - \tau y_0 - \frac{1}{2}\tau^2 x_0$$

$$y_1 = y_0 + \frac{1}{2}\tau(\alpha_0 x_0 + \alpha_0 x_1) \tag{86}$$

Substitution of $x_1 = zx_0$, $y_1 = zy_0$ in Equation (86) results in the system

$$\begin{bmatrix} z - 1 + \frac{1}{2}\tau^2 & \tau \\ -\frac{1}{2}\tau(a_0 + a_1 z) & z - 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \tag{87}$$

from which we obtain the quadratic characteristic equation

$$\det\begin{bmatrix} z-1+\frac{1}{2}\tau^2 & \tau \\ -\frac{1}{2}\tau(a_0+a_1 z) & z-1 \end{bmatrix} = 0 \tag{88}$$

$$z^2+2\left(-1+\frac{1}{4}\tau^2+\frac{1}{4}a_1\tau^2\right)z+1+\frac{1}{2}\tau^2(a_0-1)=0$$

for magnification factor z. To assure $|x| = 1$ for the complex roots of Equation (88) we set a_0 = 1 and are left with

$$z^2+2\left(-1+\frac{1}{4}\tau^2\beta\right)z+1=0 \tag{89}$$

where $\beta = 1 + \alpha_1$. The two roots of Equation (89) are

$$z=1+\frac{1}{4}\tau^2\beta \pm i\tau\sqrt{\frac{1}{2}\beta-\frac{1}{16}\tau^2\beta^2} \tag{90}$$

and z is complex if $$\beta > 0\,,8-\tau^2\beta > 0 \tag{91}$$

Because $|x| = 1$ we may write the complex roots of Equation (89) as

$$z=\mathrm{Cos}(\theta)\pm i\,\mathrm{Sin}(\theta),\ \mathrm{Cos}(\theta)=1-\frac{1}{4}\tau^2\beta\ ,\mathrm{Sin}(\theta)=\tau\sqrt{\frac{1}{2}\beta-\frac{1}{16}\tau^2\beta^2} \tag{92}$$

The numerical scheme is period conserving if $\tau = \theta$, or $\sin\theta = \sin\tau$. This is assured, according to Equation (92), by β such that

$$\mathrm{Sin}(\tau)=\tau\sqrt{\frac{1}{2}\beta-\frac{1}{16}\tau^2\beta^2} \tag{93}$$

$$\frac{1}{16}\tau^2\beta-\frac{1}{2}\beta+\left(\frac{\mathrm{Sin}\,(\tau)}{\tau}\right)^2=0 \tag{94}$$

Resulting in

$$\beta=4(1-\mathrm{Cos}(\tau))/\tau^2 \tag{95}$$

$$\alpha_1=1-\frac{1}{6}\tau^2+\frac{1}{180}\tau^4 \tag{96}$$

If τ is small

13. Conclusion

We accomplished here showing how to routinely determine the consistency and stability of any multistep method, explicit as well as implicit, for the stepwise integration of the first order initial value problem. We have also demonstrated here the advantage of using implicit methods to capture different solutions emanating from a branchoff point. For the integration of the second order equation of motion we have shown how to advance and retard the motion of the computed solution.

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