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Analytical Algorithm for Systems of Neutral Delay Differential Equations

Noorhabib Nazarkhil

Research Scholar Associate Professor, Department of Mathematics, Kandahar University (Afghanistan)

Abstract

Delay differential equations (DDEs), as well as neutral delay differential equations (NDDEs), are often used as a fundamental tool to model problems arising from various areas of sciences and engineering. However, NDDEs particularly the systems of these equations are special transcendental in nature; it has therefore, become a challenging task or times almost impossible to obtain a convergent approximate analytical solution of such equation. Therefore, this study introduced an analytical method to obtain solution of linear and nonlinear systems of NDDEs. The proposed technique is a combination of Homotopy analysis method (HAM) and natural transform method, and the He's polynomial is modified to compute the series of nonlinear terms. The presented technique gives solution in a series form which converges to the exact solution or approximate solution. The convergence analysis and the maximum estimated error of the approach are also given. Some illustrative examples are given, and comparison for the accuracy of the results obtained is made with the existing ones as well as the exact solutions. The results reveal the reliability and efficiency of the method in solving systems of NDDEs and can also be used in various types of linear and nonlinear problems.

Keywords: *Homotopy Analysis Method, He's Polynomial and Neutral Delay Differential Equations, Natural Transform.*

1. Introduction

Ordinary differential equations (ODEs) are usually used as a fundamental tool in modelling the problems of the real world. However, in most cases, the mathematical formulation of real-life problems needs to consider both the present and past states of the system behavior. Different from (ODEs), DDE is a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous time [1]. Hence, more reliable models of real problems arising from various field of studies such as; biology, population dynamics, chemistry, and physics, control theory to mention but few are now model using DDEs as well as NDDEs [2] [3] [4]. Recently series of methods have been developed to find an approximate analytical solution to different types of DDEs [5] [6]. However, most of these methods have experienced a series of challenges in finding a convergent approximate analytical solution of NDDEs in particular system of such equations. So, scientist and engineers adopt the use of numerical methods as the best approach to approximate the solutions. Therefore, more analytical approaches are highly needed for solving these equations.

This work aims to develop an analytical technique for solving linear and nonlinear systems of NDDEs from the combination of HAM and Natural transform. The proposed technique improved on the work of Rebenda and Smarda [7] by introducing the concept of NDDE into Natural transform. In addition, the various derivatives for both proportional and constants delay of NDDEs were successfully generated using the Natural transform. This work is also an extension of Efficient Analytical Approach for Nonlinear System of Retarded Delay Differential equations. This approach was developed by Barde and Maan [8] and solutions to different types of nonlinear systems of retarded DDEs were obtained in a series form which converges to exact or approximate solution.

Thus, based on the results of this pervious works, this research focuses to develop a new analytical technique that modifies Efficient Analytical Approach for Nonlinear System of Retarded Delay Differential Equations with aims to obtain approximate analytical solution for both linear and nonlinear system of NDDEs with proportional and constant delays. Using the introduced technique, the He's polynomial is adjusted in order to ease the computational difficulties of nonlinear terms of such equations. Furthermore, the convergence analysis and the maximum estimated error of the technique are also investigated.

Therefore, in this work we were able to develop a new generating function in Equation (29) that provides a convergent analytical solution to various types of linear and nonlinear system of NDDEs in a series form using few numbers of computational terms and minimal error as compared with the previous techniques. Thus, different from some of the existing methods the presented technique provides solution to different form of linear and nonlinear NDDEs without any linearization, perturbation or unnecessary assumptions. In Section 4, some illustrative examples are presented in order to show the reliability and efficiency of our algorithms over the reference methods.

2. Methods

The idea of this work is come up with analytical approach from the combination of natural transform and HAM for solving systems of linear and nonlinear NDDEs. The He's polynomial is modified to compute the series of nonlinear terms of both proportional and constants delays.

HAM is a powerful technique introduced by Lio [9] for solving different types of linear and nonlinear problems. Details on theory and application of HAM can be found in [1] [10] [11].

In recent years natural transform is considered as an active topic in research due to its vast application in solving different type of differential and integral equations [12] [13] . This transform was derived from the renowned

Fourier integral which converged to either Laplace or Sumudu transforms depending on the values of the transform variables. The basic concepts of natural transform for further use in this research are rendered below.

Definition 2.1 [14] Let $\tau \in (-\alpha, \alpha)$ then the natural transform of the function v (t) is defined by:

$$N^+[v(\tau)] = V(s,u) = \int_{-\alpha}^{0} e^{-st} v(ut)\,dt; \qquad s,u \in (0,\alpha) \tag{1}$$

Where N^+ denotes as natural transform and s, u are transforming variables. Equation (1) can be simplified as [14]

$$\begin{aligned} N^+[v(\tau)] = V(s,u) &= \int_{-\alpha}^{0} e^{-st} v(ut)\,dt; \qquad s,u \in (0,\alpha) \\ &= \int_{-\alpha}^{0} e^{-st} v(ut)\,dt;\ \ s,u \in (-\alpha,0) + \int_{0}^{\alpha} e^{-st} v(ut)\,dt;\ \ s,u \in (0,\alpha) \\ &= N^-[v(\tau)] + N^+[v(\tau)] \qquad (2) \\ &= N[v(\tau)H(-\tau)] + N[v(\tau)H(\tau)] \\ &= V^-(s,u) + V^+(s,u) \end{aligned}$$

Where, H (.) is the Heaviside function. Assume the function $v(\tau)H(\tau)$ is defined on $R^+$and for t ∈R then its natural transform can be define over the set

$$A = \left\{ v(\tau) : \exists M, \tau_1 \tau_2 > 0, |v(\tau)| < Me^{\frac{|\tau|}{\tau j}}, \tau \in (-1)^j \times [0,\alpha), j \in Z^+ \right\}$$

As in the given integral:

$$N^+[v(\tau)] = V^+(s,u) = \int_{0}^{\alpha} e^{-st} v(ut)\,dt; \ \ s,u \in (0,\alpha] \tag{3}$$

Theorem 2.1 [15] the generalized natural transform of the function $v(\tau)$ is given as

$$N^+[v(\tau)] = V^+(s,u) = \sum_{n=0}^{\alpha} \frac{n!\,\alpha_n u^n}{s^{n+1}} \tag{4}$$

Property 2.1 [14] Let a be a non-zero constant and $v(\alpha\tau)$ ∈A then,

$$N^+[v(\alpha\tau)] = \frac{1}{a} V\left(\frac{s}{a}, u\right) \tag{5}$$

Theorem 2.2 [13] If H_τ is the Heaviside function and for any real number $\tau \geq 0$ we defined

$$H_\tau(\tau) = \begin{cases} 1, & \text{for } t \geq \tau \\ 0, & \text{for } t < \tau \end{cases}$$

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Then the natural transform of the shifted function $v(t-\tau)-v(t-\tau)H_\tau$ is given by

$$N^+[v(t-\tau)H_\tau(t)] = e^{\frac{-s\tau}{u}}N^+[v(\tau)] \tag{6}$$

Theorem 2.3 [15] Let $v^{(n)}(t)$ be the nth derivatives of the function $v(t)$ then its natural transform is given by

$$N^+[v^n(t)] = v_n^+(s,u) - \sum_{k-1}^{n}\frac{s^{n-k}}{(n-k)+1}(0) \tag{7}$$

Corollary 2.1 Let $v_i^{(n)}(at)$ be the nth derivatives of the functions $v_i(at)$ with respect to $t(i=1,2\dots,N)$ and suppose that $N^+[v_i(at)] = v_i^+(s,u)$ then we define the following

$$N^+[v_i^n(at)] = v_{n,i}^+(as,u) = \frac{s^n}{(au)^n}v_i^n(as,u) - \sum_{k-1}^{n}\frac{s^{n-k}}{(n-k)+1}v_i^{(k-1)}(0) \tag{8}$$

Corollary 2.2 Suppose $v_i^n(t-\tau)$ are the nth derivatives of the shifted functions $v_i(t-\tau)$ with respect to t, then their natural Transforms can be define as

$$N^+[v_i^n(t-\tau)] = e^{\left(\frac{s\tau}{u}\right)}v_{n,i}^+(as,u) \tag{9}$$

$$= \frac{s^n}{u^n}e^{\left(\frac{s\tau}{u}\right)}v_i^+(s,u) - \sum_{k-1}^{n}\frac{s^{n-k}}{(u)^{(n-k)+1}}v_i^{(k-1)}\left[\lim_{i\to 0}v_i^{*k-1)}(t-\tau)\right]$$

3. Analysis of the Result

Consider the following n-order system of NDDEs

$$[v_i(t)+v_i(\alpha(t)]^n = F_i\left[t, v_\gamma^{(p)}(t), v_\gamma^{(p)}(\alpha_{i,j}(t))\right], t\in[0,d], i=1,\dots,N, J=1,\dots M \tag{10}$$

$$v_\gamma^{(p)}(t) = (v_1^{(p)}, v_2^{(p)},\dots,v_N^{(p)}$$

$$v_\gamma^{(p)}\left(\alpha_{i,j}(t)\right) = \left(v_1^{(p)}(\alpha_{i,j}(t)\right), v_2^{(k)}\left(\alpha_{i,j}(t)\right),\dots,v_N^{(p)}\left(\alpha_{i,j}(t)\right)$$

For $p=0,1,2,\dots.n-1$ and $\alpha_{i,j}(t)$ are the functions of delay terms such that $\alpha(t) = \max[\alpha_{i,j}(t)]$ with the given initial conditions

$$v_i^{(p)}(0) = v_{i,0}^{(p)}, \quad v_i(t) = \Psi_i(t), \quad t<0 \tag{11}$$

Now for simplicity we rewrite Equation (14) in the following form

$$L_i(v_i + v_i(\alpha)) + R_i(v_\gamma) + F(v_\gamma) = g_i(t) \tag{12}$$

Subject to a given initial conditions. The v_γ is defined as N-dimensional vector of the form $v_\gamma = [v_1(t), v_2(t), \dots, v_N(t)]$. The linear terms are decomposed into bounded linear operators L_i , R_i (That is there are some positive numbers $\alpha_{i,1}$, $\alpha_{i,2}$ such that $\{\|L_i(v_\gamma)\| \le \alpha_{i,1}\|v_\gamma\|, \|R_i(v_\gamma)\| \le \alpha_{i,2}\|v_\gamma\|\}$ with L_i as the highest order and R_i as remaining of the linear operators, and F_i are continuous functions satisfy the Lipschitz condition with Lipschitz constants $\mu_i \in [0, d](|f_i(v) - f_i(u)| < \mu_i|v - u|, \forall t \in [0, d])$ represent the nonlinear terms.

Take the natural transform of both sides of Equation (12) to obtain:

$$N^+[L_i(v_i + v_i(\alpha_i))] + N^+[R_i(v_\gamma)] + N^+[F_i(v_\gamma)] = N^+[g_i(t)] \tag{13}$$

Note: This research considered two forms of delay functions $\alpha_{i,j}(t)$ as follows:

Case I: $\alpha_{i,j}(t) = a_{i,j}t$, where $\alpha_{i,j} \in (0,1)$ (proportional delay).

Case II: $\alpha_{i,j}(t) = t - \tau_{i,j}$, where $\tau_{i,j} > 0$ are real constants (constant delay) Therefore, by substituting the given initial condition into Equation (13), and simplify using the differential properties of natural transform we respectively obtained the following for the two types of delay as defined in Case I and Case II.

$$N^+\left[v_i(t) + \frac{1}{a_i^{(n+1)}} v(\alpha_i)\right] - \sum_{k=1}^{n}\left(1 + \frac{1}{a_i^{(n+1-k)}}\right)\frac{u^{n-k}}{s^k} v_i^{k-1}(0)$$

$$+\frac{u^n}{s^n} N^+[R_i(v_\gamma) + F_i(v_\gamma) - g_i(t)] = 0 \tag{14}$$

$$N^+\left[v_i(t) + e^{\frac{-s\tau_i}{u}}(t)\right] - \sum_{k=1}^{n}\frac{u^{n-k}}{s^k}[v_i^{k-1}(0) + \lim_{t\to 0} v_i^{k-1}(t-\tau_i)]$$

$$+\frac{u^n}{s^n} N^+[R_i(v_\gamma) + F_i(v_\gamma) - g_i(t)] = 0$$

Where $a_i = \max[a_{i,j}]$ and $\tau_i = \max[\tau_{i,j}]$.Now from Equation (14) we can define the following nonlinear operators

$$N_i[\phi_i(t;q)] = N^+[\phi_i(t;q) + \frac{1}{a_i^{(n+1)}}\phi_i(a_i(t;q)] - \sum_{k=1}^{n}\left(1 + \frac{1}{a_i^{(n+1-k)}}\right)\frac{u^{n-k}}{s^k}\phi_i^{k-1}(9)$$

$$+\frac{u^n}{s^n} N^+\left[R_i\left(\phi_\gamma(t;q)\right) + F_i\left(\phi_\gamma(t;q)\right) - g_i(t)\right] \tag{15}$$

$$N_i[\phi_i(t;q)] = N^+\left[\phi_i(t;q) + e^{\frac{-s\tau_i}{u}}\phi_i(t;q)\right] - \sum_{k=1}^{n}\frac{u^{n-k}}{s^k}\left[\phi_i^{k-1}(0) + \lim_{t\to 0}\phi_i^{k-1}(t;q-\tau_i)\right]$$

$$+\frac{u^n}{s^n}N^+\left[R_i\left(\phi_\gamma(t;q)\right) + F_i\left(\phi_\gamma(t;q)\right) - g_i(t)\right]$$

Where $q \in [0,1]$ is an embedding parameter, $\phi_i(t;q)$ are functions of variables t and q. So, by means of HAM we can construct the following Homotopy Equations as

$$(1-q)N^+\left[\phi_i(t;q) - v_{i,0}(t)\right] = h_i q H_i(t)\, N_i[\phi_\gamma(t;q)] \quad (16)$$

Where N^+ denotes as natural transform, $v_{i,0}(t)$ are initial approximations of $v_i(t)$ and $h_i H(t)$ are non-zero auxiliary parameters and auxiliary functions respectively.

Now, from Equation (16) as q=0 and q=1 we respectively obtained the following equation.

$$\phi_i(t,0) = v_{i,0}(t)$$

$$\phi_i(t,1) = v_i(t) \quad (17)$$

Thus, as q increases from 0 to 1, the solutions $\phi_i(t,q)$ vary from the initial approximations $v_{i,0}(t)$ to the exact solutions$v_i(t)$. In topology, this type of variation is called deformation and Equation (16) is called zero-order deformation equation. Therefore, the Taylor series expansion of $\phi_i(t,q)$,with respect to q can be obtained as

$$\phi_i(t,q) = \phi_i(t,0) + \sum_{m=1}^{\alpha} v_{i,m}(t)q^m \quad (18)$$

$$v_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t,q)}{\partial q^m}\int_{q=0}$$

Suppose that the initial approximations $v_{i,0}$(t) auxiliary parameters h_i and the auxiliary function $H_i(t)$ are properly chosen so that the series in Equation (18) converges at q =1, that is

$$\phi_i(t,1) = v_{i,0}(t) + \sum_{m=1}^{\alpha} v_{i,m}(t) \quad (19)$$

Define vectors

$$v_{i,n}(t) = [v_{i,0}(t), v_{i,1}(t), \ldots, v_{i,n}(t)] \quad (20)$$

By differentiate Equation (16) m times with respect to q and setting q=0 and finally divided by m! obtain the so called mth-order deformation equation as

$$N^+\left[v_{i,m}(t) - x_m v_{i,m-1}(t)\right] = h_i H_i(t) R_{y_i m}[v_{i,m-1}(t)] \quad (21)$$

$$R_{y_i m}\left[v_{i,m-1}(t)\right] = \frac{1}{(m-1)!}\frac{\partial^{m-1}N_i[\phi_i(t,q)]}{\partial q^{m-1}}\int_{q=0} \qquad (22)$$

$$x_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

By taking the inverse natural transform on both sides of Equation (21) we obtained

$$v_{i,m}(t) = x_m v_{i,m-1}(t) + h_i N^{+}[H_i(t) R_{y_i m}\left[v_{i,m-1}(t)\right]] \qquad (23)$$

Therefore, $v_{i,m}(t)$ for $m \geq 1$ can be easily obtained from Equation (23), at Mth order we have

$$v_i(t) = \sum_{m=0}^{m} v_{i,m}(t) \qquad (24)$$

Hence, as M →∞ the following recursive relations of Equations (10) and (11) for the two type of delay as defined respectively in Case I and Case II are obtained

$$v_{i,m}(t) = (x_m + h_i)v_{i,m-1}(t) + h_i \frac{1}{a_i^n} v_{i,m-1}(a_i t)$$

$$-h_i(1 - x_m)N^{-}\sum_{k=1}^{n}\left(1 + \frac{1}{a_i^{(n-1+k)}}\right)\frac{u^{k-1}}{s^k}\phi_i^{k-1}(0)$$

$$+h_i N^{-}\left\{\frac{u^n}{s^n}N^{+}\left[R_i(v_{\gamma,m-1}(t)\right) + H_{i,m-1}(v_{\gamma 1}, v_{\gamma 2}, \dots, v_{\gamma N}) - g_i(t)\right\}, m \geq 1 \qquad (25)$$

$$v_{i,m}(t) = (x_m + h_i)v_{i,m-1}(t) + h_i v_{i,m-1}(t - \tau_i)$$

$$-h_i(1 - x_m)N^{-}\sum_{k=1}^{n}\frac{u^{k-1}}{s^k}[v_i^{k-1}(0) + \lim_{t\to 0} v_i^{k-1}(t - \tau_i)]$$

$$+h_i N^{-}\left\{\frac{u^n}{s^n}N^{+}\left[R_i(v_{\gamma,m-1}(t)\right) + H_{i,m-1}(v_{\gamma 1}, \dots, v_{\gamma N}) - g_i(t)\right\}, m \geq 1$$

Now, the nonlinear operators $F_i(v_\gamma)$ are expanded as series of modified He's polynomials $H_{i,m-1}(v_{\gamma 1}, v_{\gamma 2}, \dots, v_{\gamma N})$ define as

$$H_{i,m-1}(v_{\gamma 1}, v_{\gamma 2}, \dots, v_{\gamma N}) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}F_i\left(\sum_{p=0}^{m} q^p v_{\gamma,p}\right)\int_{q=0} \qquad (26)$$

Where $v_{\gamma,i} = (v_{i,1}, v_{i,2}, \dots, v_{i,N})$ and $v_{\gamma,p} = (v_{1,p}, v_{2,p}, \dots, v_{N,p})$. The proof for Case II (constant delay) is of the same process with that of Case I.

Theorem 3.1 Assume $(C[D], \|.\|)$ is a Banach Space and let $v_{\gamma,m}(t)$ be defined in $(C[D], \|.\|)$, where $\|v_{\gamma,m}\|$ are define in form of an operators, that is $v_{\gamma,m}(t) = A_i(v_{\gamma,m-1}(t))$ such that $A_i(v) = -N^-[\vec{v_{i,m-1}}(t)]$ and

$$\|A_i(v) - A_i(u)\| \leq \delta_i \|u - v\|, \quad \forall v, u \in C[D] \tag{27}$$

Where $\delta_i = (\alpha_{i,1} + \alpha_{i,2} + \mu_i)d$ for some $\delta_i \in (0,1)$. Then A_i have unique fixed points in C[D]. Furthermore, the Homotopy series in Equation (19) converged uniquelly to the solutions $v_i(t)$ (their respective fixed points in C[D]) of Equations (10) and (11).

Theorem 3.2 Suppose the Homotopy series in Equation (19) converges to the solution $v_i(t)$ of Equations (10) and (11) and let the approximations of $v_i(t)$ are given by the truncuted series $\sum_{p=0}^{m} v_{i,m}(t)$. Then the maximum absolute error is estimated to be

$$\left\| v_i(t) - \sum_{p=0}^{m} v_{i,m}(t) \right\| \leq \frac{\delta_i^n}{1-\delta_i} \|v_i(t)\| \tag{28}$$

4. Examples and Discussion

The application of the proposed technique will be presented in this section. This involves solving some problems of linear and nonlinear systems of NDDEs with both proportional and constant delays.

Example 4.1 [16] First we seek for a solution of the following 2-dimensional linear system of NDDEs with constant delay

$$v_1'(t) = v_1'(t-1) + 4\, v_2(t), \qquad 0 \leq t \leq 2 \tag{29}$$

$$v_2'(t) = v_1(t) - v_1(t-1), \qquad 0 \leq t \leq 2$$

$$v_1(t) = e^{-2t},\ v_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right), \qquad t \in [-1,0].$$

Take the natural transform to both sides of Equation (29) and simplify further using Equation (9) to get

$$N^+\left[v_1(t) - e^{-\frac{s}{u}} v_1(t)\right] - \frac{1}{s}\left[1 - e^2\right] - \frac{u}{s} N^+[\,4v_2(t)] = 0 \tag{30}$$

$$N^+[v_2(t)] - \frac{1}{s}\left[\frac{1}{2}(e^2 - 1)\right] + \frac{u}{s} N^+[\,v_1(t-1) - v_1(t)] = 0$$

From Equation (30) define a non-linear operator

$$N[\phi_1(t;q)] = N^+\left[\phi_1(t;q) - e^{-\frac{s}{u}}\phi_1(t;q)\right] - \frac{1}{s}\left[1-e^2\right] - \frac{u}{s}N^+[4\phi_2(t;q)] \tag{31}$$

$$N[\phi_2(t;q)] = N^+\left[\phi_2(t;q)\right] - \frac{1}{s}\left[\frac{1}{2}(e^2-1)\right] + \frac{s}{u}N^+[\phi_1(t-1;q) - \phi_1(t;q)]$$

Now using Equation (25) the recursive relation of Example 4.1 can be obtained as

$$v_{1,m}(t) = (x_m + h_1)v_{1,m-1}(t) - h_1 v_1(t-1) - h_1(1-x_m)N^-[\tfrac{1}{s}(1-e^2)]$$

$$-h_1 N^-\left\{\frac{u}{s}N^+[R_1(v_{2,m-1}(t))]\right\} \tag{32}$$

$$v_{2,m}(t) = (x_m + h_2)v_{2,m-1}(t) - h_2(1-x_m)N^-[\tfrac{1}{2s}(e^2-1)]$$

$$+h_2 N^-\left\{\frac{u}{s}N^+[R_2(v_{1,m-1}(t))]\right\}, \quad m \geq 1$$

By choosing an initial approximations of $v_{1,0}(t) = 1$ and $v_{2,0}(t) = \frac{1}{2}(e^2-1)$ and using Equation (32) we obtained the following

$$v_{1,1}(t) = -h_1(e^2-1)t\,,\ v_{2,1}(t) = h_2(e^2-1)t$$

$$v_{1,2}(t) = -h_1h_2(e^2-1)t^2 - 2h_1(e^2-1)t$$

$$v_{2,2}(t) = (h_2^2 + 2h_1h_2 + h_2)(e^2-1)t$$

$$v_{1,3}(t) = (h_1h_2^2 - 3h_1h_2)(e^2-1)t^2 - 2h_1(e^2-1)t \tag{33}$$

$$v_{2,3}(t) = -\frac{2}{3}h_1h_2(e^2-1)^2t^3 - (2h_1h_2^2 + h_1h_2)(e^2-1)^2t^2$$

$$+(h_2^3 + 2h_2^2 + h_2)(e^2-1)t$$

Following the same process remaining terms of $v_{i,m}(t)$ for $m \geq 3$ can be obtained. Putting $h_1 = \frac{1}{3(e^2-1)}$ and $h_2 = -1$ in Equation (33), then the fifth order approximation of Example 4.1 is given as

$$v_1(t) = 1 - 2t + 2t^2 - \frac{4}{3}t^3 + \frac{2}{3}t^4 - \frac{4}{15}t^5 + \frac{4}{45}t^6 + \cdots$$

$$v_2(t) = \frac{1}{2}(e^2-1) - (e^2-1)t + (e^2-1)t^2 - \frac{2}{3}(e^2-1)t^3 \tag{34}$$

$$+\frac{1}{3}(e^2-1)t^4 - \frac{2}{15}(e^2-1)t^5 + \frac{2}{45}(e^2-1)t^6 + \cdots$$

The series solutions in Equation (34) converged to exact solutions

$$v_1(t) = e^{-2t}, v_2(t) = \frac{1}{2}\left(e^{-2(t-1)} - e^{-2t}\right)$$ of Equation (29).

Therefore, the fifth order approximated series of the derived algorithm in Equation (25) was successfully generates the closed form solution of Example 4.1 with minimum error as shown in Figure 1 While in most applications only numerical approximations was obtained. For instance, in [16] the numerical approximation of this problem was computed using Implicit Block method with the maximum absolute error of 1.47320 when the tolerance was 1×10^{-10} in a total number of 25 steps.

Example 4.2 [7] Next we consider a third-order nonlinear system of NDDEs with both proportional and constant delays

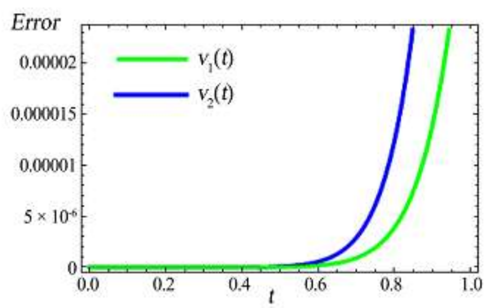

Figure 1.The behavior of maximum absolute errors between the exact solution and fifth-order approximation of Example 4.1.

$$v_1'''(t) = v_1'''(t-1)v_1\left(\frac{t}{3}\right) + (v_1(t))^{\frac{2}{3}} + 2t + e^{-t} \quad (35)$$

$$v_2'''(t) = \frac{1}{2}v_2'''\left(\frac{t}{2}\right) + v_2'(t-1)v_1\left(\frac{t}{3}\right), \ t \geq 1$$

With initial functions

$\phi_1(t) = e^t, \phi_2(t) = t^2 \ , t \in [-2,0]$ And the given initial conditions

$v_1(0) = 1, v_1'(0) = 1, \quad v_1''(0) = 1$

$v_2(0) = 0, v_2'(0) = 0, \quad v_2''(0) = 2$

Take the natural transform to both sides of Equation (35) and simplify further using Equation (9) to get

$$N^+[v_1(t)] - \left[\frac{1}{s} + \frac{u}{s^2} + \frac{u^2}{s^3}\right] - \frac{u^3}{s^3} N^+\left[e^{t-2} v_1\left(\frac{t}{2}\right) + (v_1(t))^{\frac{2}{3}} + 2t + e^{-t}\right] = 0 \tag{36}$$

$$N^+\left[v_2(t) - 8v_2\left(\frac{t}{2}\right)\right] - \frac{u^3}{s^2} N^+\left[2(1-t) v_1\left(\frac{t}{3}\right)\right] = 0$$

From Equation (36) define a non-linear operator

$$N[\phi_1(t;q)] = N^+[\phi_1(t;q)] - \left[\frac{1}{s} + \frac{u}{s^2} + \frac{u^2}{s^3}\right]$$

$$-\frac{u^3}{s^3} N^+\left[e^{t-2}\phi_1\left(\frac{t}{2};q\right) + (\phi_1(t;q))^{\frac{2}{3}} + 2t + e^{-t}\right] \tag{37}$$

$$N[\phi_2(t;q)] = N^+\left[\phi_2(t;q) - 8\phi_2\left(\frac{t}{2};q\right)\right] - \frac{u^3}{s^2} N^+\left[2(1-t)\phi_1\left(\frac{t}{3};q\right)\right] = 0$$

Now using Equation (25) the recursive relation of Example 4.2 can be obtained as

$$v_{1,m}(t) = (x_m + h_1) v_{1,m-1} - h_1(1 - x_m) N^-\left[\frac{1}{s} + \frac{u}{s^2} + \frac{u^2}{s^3}\right]$$

$$-h_1 N^-\left\{\frac{u^3}{s^3} N^+\left[R_1\left(v_{1,m-1}(t)\right) + H_{1,m-1}(v_{\lambda_1}, \dots, v_{\lambda_N}) + g_1(t)\right]\right\} \tag{38}$$

$$v_{2,m}(t) = (x_m + h_2) v_{2,m-1}(t) - 4h_2 v_{m-1}\left(\frac{t}{2}\right) - h_2 N^-\{\frac{u^3}{s^3} N^+[R_2\left(v_{2,m-1}(t)\right)$$

By choosing an initial approximations of $v_{1,0}(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$ and $v_{2,0}(t) = t^2$ and using Equation (38) we obtained the following

$$v_{1,1}(t) = -\left(\frac{1+e^{-2}}{6}\right) h_1 t^3 - \left(\frac{5+4e^{-2}}{72}\right) h_1 t^4 - \left(\frac{16e^{-2}-5}{1080}\right) h_1 t^5 + \left(\frac{1098e^{-2}+506}{1145}\right) h_1 t^6 \tag{39}$$

$$v_{2,1}(t) = \frac{1}{3} h_2 t^3 - \frac{1}{27} h_2 t^4 - \frac{1}{108} h_2 t^5 + \cdots$$

By putting $h_1 = -1$ and $h_2 = -2$ in the series Equation (39) we obtained the pproximate solution of Example 4.2 as

$$v_1(t) = 1 + t + \frac{t^2}{2} + \left(\frac{1+e^{-2}}{6}\right) t^3 + \left(\frac{5+4e^{-2}}{72}\right) t^4 + \left(\frac{16e^{-2}-5}{1080}\right) t^5 - \left(\frac{1098e^{-2}+506}{1145}\right) t^6 \tag{40}$$

$$v_2(t) = t^2 - \frac{2}{3} t^3 + \frac{2}{27} t^4 + \frac{1}{54} t^5 + \cdots$$

Using only one iteration of the derived algorithm (first order) in Equation (25) a good approximation of Example 4.2 was successfully obtained.Since this problem has no exact solution therefore, Figure 2 shows the comparison between approximate solution obtained by the proposed technique, Matlab Package DDENSD and the result obtained by Rebenda and Smarda [7] using an algorithm based on the combination of the method of steps and differential trans form method (DT).

Threrefore, from Figure 2 we can observed that there is a good correspon-dence between the first order approximate solution of the proposed tecnique with that of Matlab Package DDENSD and DT. Hence, this shows that the presented method provides reliable results and reduces the computation size as compared with the previous techniques.

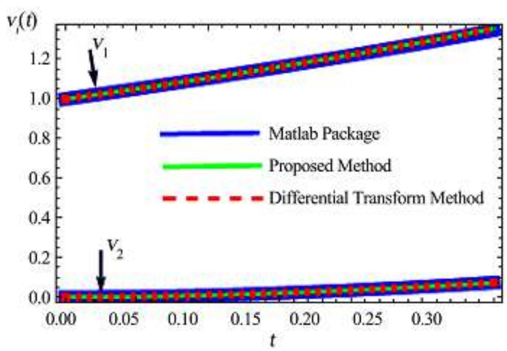

Figure 2. Comparison of solutions obtained by Matlab Package, Proposed Method and Differential Transform Method of Example 4.2.

5. Conclusion

This paper presents an efficient analytical approach suitable for solving linear and nonlinear systems of NDDEs with proportional and constant delays via HAM and natural transform. The presented algorithm adjusted the He's polynomial in order to ease the computational difficulties of both proportional and constant delays. Another advantage of this research is that a new algorithm is constructed in Equation (21) which reduces the computational work as compared to other methods, produces a much faster convergent approximate solution and handles more complicated problems (as in the case of second example) in applications than other analytical methods. Therefore, the presented approach is efficient and reliable in solving different form of linear and nonlinear systems of NDDEs which can be also applied to solve various types of linear and nonlinear problems.

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